

Vector Spaces

We have now seen a couple of examples of vector spaces, but we haven't mentioned this explicitly. In this section, we will.

Example 1: \mathbb{R}^2 = the set of all two dimensional vectors with real entries.

Earlier in the semester we saw that this set had several properties. Addition was defined, scalar multiplication was defined, and we had associativity of addition.

Example 2: Similarly for \mathbb{R}^n , the set of all n -dimensional vectors with real entries.

Example 3: $\mathbb{R}^{m \times n}$ = the set of $m \times n$ matrices with real entries.

Now that we have recalled several sets with similar properties, let's generalize them.

Definition: Let V be a set on which two operations (usually denoted by addition and scalar multiplication) are defined. The set (together with the operations) is called a **vector space** if the following axioms are satisfied:

- (1) $x + y = y + x$ for all $x, y \in V$.
- (2) $(x + y) + z = x + (y + z)$ $x, y, z \in V$.
- (3) there exists an element, usually denoted by $0 \in V$, such that $x + 0 = x$ for all $x \in V$.
- (4) for every $x \in V$ there exists an element, usually denoted by $-x \in V$, such that $x + (-x) = 0$.
- (5) $\alpha(x + y) = \alpha x + \alpha y$ for all $x, y \in V$ and scalar α .
- (6) $(\alpha + \beta)x = \alpha x + \beta x$ for all $x \in V$ and scalars α, β .
- (7) $(\alpha\beta)x = \alpha(\beta x)$ for all $x \in V$ and scalars α, β .
- (8) $1x = x$ for all $x \in V$.

Note: Implied in this definition is that the set V must be closed under vector addition and scalar multiplication. Also, the set of scalars will be assumed to be

the real numbers unless otherwise stated. (For those with some exposure to modern algebra: In general, the set of scalars can come from any field F , in which case we say that V is a **vector space over F** . If the set of scalars is only a ring, then the set of “vectors” is called a **module**.)

Example 4: Let P_n denote the set of all polynomials of degree less than n .

Our two operations are addition and scalar multiplication (as usual). Notice that the sum of two polynomials of degree less than n also has a degree less than n . Similarly, multiplying a polynomial by a scalar can only reduce its degree. The zero polynomial is the identity. For $p(x) \in P_n$, its inverse is $-p(x)$.

Example 5: Let $C[a,b]$ denote the set of all continuous real-valued functions that are defined on $[a,b]$.

Our two operations are again addition and scalar multiplication. Since the sum of two continuous functions is still continuous (and the same hold for scalar multiples), this set is closed under these operations. The zero function is continuous and is therefore the identity. For $f \in C[a,b]$, its inverse is $-f$.

Definition: If S is a non-empty subset of a vector space V and S is also closed under the two operations of V , then S is a **subspace** of V . Every non-trivial vector space V has at least two subspaces; namely $\{0\}$ and V . Any other subspace is called a **proper subspace**.

Example 6: P_n is a proper subspace of $C[\mathbb{R}]$ for any n .

Clearly $P_n \neq \emptyset$ for any n . Since polynomials are continuous and defined everywhere, $P_n \subseteq C[\mathbb{R}]$. Since there exist continuous functions on \mathbb{R} that are not polynomials (e.g. $f(x) = \cos(x)$), $P_n \neq C[\mathbb{R}]$. So P_n is a proper subspace.

Example 7: Let $X = \{(x_1, x_2)^T : x_1 = 3x_2\}$. Is X a subspace of \mathbb{R}^2 ?

Let $\begin{bmatrix} 3b \\ b \end{bmatrix}$ and $\begin{bmatrix} 3y \\ y \end{bmatrix}$ be two arbitrary elements of X . Then $\begin{bmatrix} 3b \\ b \end{bmatrix} + \begin{bmatrix} 3y \\ y \end{bmatrix} = \begin{bmatrix} 3b + 3y \\ b + y \end{bmatrix} = \begin{bmatrix} 3(b + y) \\ b + y \end{bmatrix}$ and $\alpha \begin{bmatrix} 3b \\ b \end{bmatrix} = \begin{bmatrix} 3\alpha b \\ \alpha b \end{bmatrix}$ are clearly still in X .

Definition: The *nullspace* of an $m \times n$ matrix A is the set

$$N(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}.$$

Example 8: Find the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

If we rewrite the matrix equation $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a system, we get

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 4x_2 &= 0 \end{aligned}$$

Solving this system, we see that $x_1 = 0, x_2 = 0$ is the only solution. So $N(A) = \{\mathbf{0}\}$. (Alternately, we could have noted that since A is nonsingular (its determinant is -2), the only solution to the given matrix equation is the zero vector by Theorem 3 from the section on Elementary Matrices.)

Question: Is $N(A)$ a *subspace* of \mathbb{R}^n ?

Answer: If not, we named it poorly.

Theorem 1: Let $A \in \mathbb{R}^{m \times n}$. The $N(A)$ is a subspace of \mathbb{R}^n .

Proof: Let $x, y \in N(A)$ and $\alpha \in \mathbb{R}$. Then $A(x + y) = Ax + Ay = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and $A(\alpha x) = \alpha(Ax) = \alpha\mathbf{0} = \mathbf{0}$. So $x + y \in N(A)$ and $\alpha x \in N(A)$.

Definition: Let V be a vector space and let $v_1, v_2, \dots, v_n \in V$. A sum of the form $a_1v_1 + a_2v_2 + \dots + a_nv_n$ (where $a_i \in \mathbb{R}$) is called a **linear combination** of the vectors v_1, v_2, \dots, v_n . The set of all linear combinations of a set of vectors is the **span** of the vectors. This is denoted by $\text{Span}(v_1, v_2, \dots, v_n)$.

Theorem 2: Let V be a vector space and let $v_1, v_2, \dots, v_n \in V$. Then the $\text{Span}(v_1, v_2, \dots, v_n)$ is a subspace of V .

Proof: Let $x, y \in \text{Span}(v_1, v_2, \dots, v_n)$ and $\alpha \in \mathbb{R}$. So there exist $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ such that $x = a_1v_1 + a_2v_2 + \dots + a_nv_n$ and $y = b_1v_1 + b_2v_2 + \dots + b_nv_n$. Then

$$\begin{aligned} x + y &= (a_1v_1 + a_2v_2 + \dots + a_nv_n) + (b_1v_1 + b_2v_2 + \dots + b_nv_n), \text{ and} \\ &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots + (a_n + b_n)v_n \end{aligned}$$

$$\begin{aligned} \alpha x &= \alpha(a_1v_1 + a_2v_2 + \dots + a_nv_n) \\ &= (\alpha a_1)v_1 + (\alpha a_2)v_2 + \dots + (\alpha a_n)v_n. \end{aligned}$$

So $x + y \in \text{Span}(v_1, v_2, \dots, v_n)$ and $\alpha x \in \text{Span}(v_1, v_2, \dots, v_n)$.

Notice that clearly the $\text{Span}(v_1, v_2, \dots, v_n) \subseteq V$. If the $\text{Span}(v_1, v_2, \dots, v_n) = V$, then there seems to be something special about $\{v_1, v_2, \dots, v_n\}$. Indeed.

Definition: If $\text{Span}(v_1, v_2, \dots, v_n) = V$, then the set $\{v_1, v_2, \dots, v_n\}$ is a **spanning set** of V . In other words, the set $\{v_1, v_2, \dots, v_n\}$ is a spanning set of V if and only if every vector in V is a linear combination of the vectors v_1, v_2, \dots, v_n .

Example 10: Are the following sets spanning sets for \mathbb{R}^2 ?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$

(a) Can every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$? We need to find the coefficients (if they exist) for which

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

In other words, we need to solve the system

$$\begin{aligned} a_1 + 3a_2 &= x_1 \\ 2a_1 + a_2 &= x_2 \end{aligned}$$

Solving for a_1 and a_2 , we get $a_1 = -\frac{1}{5}x_1 + \frac{3}{5}x_2$ and $a_2 = \frac{2}{5}x_1 - \frac{1}{5}x_2$. So “Yes.”

(b) Can every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$? We need to find the coefficients (if they exist) for which

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

In other words, we need to solve the system

$$\begin{aligned} 2a_1 + 4a_2 &= x_1 \\ 3a_1 + 6a_2 &= x_2 \end{aligned}$$

However, this system has no solutions if $x_2 \neq \frac{3}{2}x_1$. So “No.”

Homework

1. Let V be a vector space and let $x \in V$. Prove that $0x = \mathbf{0}$.
2. Let \mathbb{C} be the set of complex numbers. Define addition on \mathbb{C} (as usual) by $(a + bi) + (c + di) = (a + c) + (b + d)i$, and define scalar multiplication by $\alpha(a + bi) = (\alpha a) + (\alpha b)i$ (where α is a real number). Show that \mathbb{C} is a vector space.
3. Let S be the set of all ordered pairs of real numbers. Define addition on S by $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, 0)$, and define scalar multiplication by $\alpha(a_1, b_1) = (\alpha a_1, \alpha b_1)$. Show that this is not a vector space. Which axioms fail?
4. Let R be the set of real numbers. Define addition by $x \oplus y = \max\{x, y\}$ (the max of the two numbers), and define scalar multiplication by $\alpha x = \alpha \cdot x$ (usual multiplication). Is R a vector space? If yes, show it. If no, which axioms fail?
5. Find the nullspace of (a) $A = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$.
6. Which of the following are subspaces of \mathbb{R}^2 ?
 - (a) $\{(x_1, x_2)^T : x_1 + x_2 = 0\}$
 - (b) $\{(x_1, x_2)^T : x_1 x_2 = 0\}$
 - (c) $\{(x_1, x_2)^T : x_1 = x_2\}$
 - (d) $\{(x_1, x_2)^T : |x_1| = |x_2|\}$
7. Which of the following are subspaces of $\mathbb{R}^{2 \times 2}$?
 - (a) all 2×2 symmetric matrices.
 - (b) all 2×2 matrices A for which $a_{12} = 1$.
 - (c) all 2×2 singular matrices.
 - (d) all 2×2 matrices B for which $b_{11} = 0$.
8. Which of the following are subspaces of P_4 ?
 - (a) all polynomials in P_4 of even degree.
 - (b) all polynomials of degree 3.
 - (c) all $p(x)$ in P_4 for which $p(0) = 0$.
 - (d) all constant polynomials.
9. Which of the following are spanning sets for \mathbb{R}^3 ?
 - (a) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T\}$
 - (b) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T, (1, 2, 3)^T\}$
 - (c) $\{(2, 1, -2)^T, (3, 2, -2)^T, (2, 2, 0)^T\}$
10. We can think of \mathbb{R} as a vector space over itself. So both the vectors and the scalars are real numbers. With this in mind, prove that there are no proper subspaces of \mathbb{R} .